7. TRANSCENDENTAL FUNCTIONS

§7.1. Exponentiation

We defined addition and multiplication for natural numbers and extended these to the set of complex numbers, ℂ. However we've defined x^m only for natural numbers. Part of the reason is that there are problems when *m* is non-real, or even a negative real.

But it is possible to provide a definition of x^z when *x* is a positive real, and *z* is any complex number. This simplest way to do this is to define e^z, for the constant e, as a sum to infinity of a certain power series, and then to define x^z as $e^{z.\log x}$.

Of course this requires defining the exponential and the log functions. We define the exponential function as a power series. What exactly *is* an infinite series? It is the sum to infinity of some sequence. So we recall the definition of an infinite sequence.

§7.2. Sequences

An **infinite sequence** of elements on a set S is a function $u:\mathbb{N}\to S$. We call $u(n)$ the **n'th term** and denote it by u_n. The sequence can be written as $u_0, u_1, \ldots$ or simply (u_n) where u_n is the *n*'th term.

We now focus on infinite sequences of complex numbers where we can develop the concept of convergence.

A sequence (a_n) of complex numbers is **bounded** if there exists a real number K such that $|a_n| \le K$ for all n.
A sequence (a_n) is **null** if for all $\varepsilon > 0$ $|a_n| < \varepsilon$ for all sufficiently large n. Clearly null sequences are bounded.
What does 'sufficiently' large mean? A more formal version of this definition is that a sequence (a_n) is bounded

$$\text{if } \forall\varepsilon[\varepsilon > 0 \rightarrow \exists N \forall n[n \ge N \rightarrow |a_n| < \varepsilon]].$$

Example 1: The sequence 1, 1/2, 1/3, 1/4, ... is null.
The sequence 1, –1, 1, –1, ... is bounded but not null.
The sequence 1, 2, 3, 4, ... is unbounded.

Theorem 1: The sum and difference of null sequences are null.
Proof: Let $\varepsilon > 0$. For some N, $|a_n|$, $|b_n| < \varepsilon/2$ for all $n \ge N$ and so $|a_n \pm b_n| \le |a_n| + |b_n| < \varepsilon$. ✋☺

Theorem 2: If (a_n) is null and (b_n) is bounded then (a_nb_n) is null.
Proof: Suppose $K > 0$ such that $|b_n| \le K$ for all n and suppose that (a_n) is null. Let $\varepsilon > 0$.

Then for some N, $|a_n| < \frac{\varepsilon}{K}$ for all $n \geq N$. Hence $|a_n b_n| < \varepsilon$.
✋☺

We say that (a_n) **converges to a limit** L if the sequence $(a_n - L)$ is null and we write $\mathbf{a_n \to L}$ or $\mathbf{\lim a_n = L}$. If it doesn't converge, we say that it **diverges**. Limits, if they exist, are unique. The limit of a sum, difference or product of two convergent sequences is the sum, difference or product of the respective limits and if (a_n) converges to a non-zero limit L then $(1/a_n)$ converges to 1/L. You should have seen all of this before.

We say that a sequence (a_n) is
increasing if $a_n \leq a_{n+1}$ for all n, and
strictly increasing if $a_n < a_{n+1}$ for all n.

Similarly we define **decreasing** and **strictly decreasing**.

A sequence (a_n) is **monotone** if it is either increasing or decreasing.

Theorem 3: A bounded increasing sequence of positive reals converges.
Proof: Suppose (a_n) is a bounded increasing sequence.

Since $\cup\{a_n \mid n \in \mathbb{N}\}$ is a non-empty initial segment of $\mathbb{Q}^+$ and is bounded above, it's a positive real number, say L.
Let $\varepsilon > 0$. Since we're working with positive reals we need to have $L - \varepsilon > 0$, so let $\varepsilon_1 = \text{MIN}(\varepsilon, L)$.
If $a_n \le L - \varepsilon_1$ for all n then:

$$\cup\{a_n \mid n \in \mathbb{N}\} \le L - \varepsilon_1, \text{ a contradiction.}$$

Hence $a_N > L - \varepsilon_1$ for some N.
For all $n \ge N$, we have $L \ge a_n \ge a_N > L - \varepsilon_1$ and so

$$|a_n - L| < \varepsilon_1 < \varepsilon.$$ ✋☺

Corollary: A bounded increasing sequence of real numbers converges.
Case 1: $a_N > 0$ for some N. The sequence $a_N, a_{N+1}, \ldots$ is an increasing sequence of positive real numbers, bounded above, and so converges to some limit L. Clearly this will be $\lim a_n$.
Case 2: $a_n < 0$ for all N. Define $b_n = 1 + (a_n - a_0)$.
Then $b_n > 0$ for all n and is bounded above.
Hence (b_n) converges and hence so does (a_n).
Corollary: A bounded decreasing sequence converges.
Proof: If (a_n) is decreasing then $(-a_n)$ is increasing.

Theorem 4 (Squeeze Law):
If $a_n \le b_n \le c_n$ and $a_n \to L$ and $c_n \to L$ then $b_n \to L$.
Proof: $|b_n - L| \le \text{MAX}(|a_n - L|, |c_n - L|)$. ✋☺

Theorem 5: If $0 < a < 1$ then (a^n) converges to 0.
If $a > 1$, (a^n) diverges.

Proof: If $0 < a < 1$ then (a^n) is decreasing and bounded below. If $a > 1$, (a^n) is unbounded. ✋☺

§7.3. Infinite Series

An **infinite series** is really just an infinite sequence, for which we consider the corresponding sequence of partial sums. When we consider it as a series, we write (a_n) as Σa_n. The sequence of **partial sums** is (s_n) where

$$s_n = a_0 + a_2 + \dots + a_n.$$

The series Σa_n is said to **converge** if the sequence (s_n) converges and the limit is written using the same notation, that is, Σa_n. Sometimes we start at $n = 1$, in which case we write $\sum_{n=1}^{\infty} a_n$. If the series doesn't converge it is said to **diverge**.

Theorem 6: If Σa_n converges then $\lim a_n = 0$.

Proof: For all n, $s_n = s_{n-1} + a_n$ and $\lim s_n = \lim s_{n-1}$. ✋☺

As usual a **geometric series** is one of the form Σa^n.

If $|a| < 1$ then Σa^n converges to $\dfrac{1}{1-a}$.

Theorem 7 (Comparison Test): If $0 < a_n \le b_n$ and Σb_n converges then Σa_n converges and $\Sigma a_n \le \Sigma b_n$.
Proof: Use the squeeze law on the partial sums. ✋☺

Theorem 8 (Ratio Test):

If $\dfrac{a_{n+1}}{a_n} < r < 1$ for all sufficiently large n then

$$\Sigma a_n \text{ converges.}$$

Proof: Use the Comparison Test with $\Sigma a_0.r^n$.

Theorem 9 (Ratio Limit Test):
If $\lim a_{n+1}/a_n = L < 1$ then Σa_n converges.
Proof: For sufficiently large n, $a_{n+1}/a_n < ½(1 + L) < 1$.
✋☺

A **power series** is one of the form $\Sigma a_n z^n$. We are mostly interested in those power series that converge for all $z \in \mathbb{C}$.

Theorem 10: The power series $\Sigma a_n z^n$ converges for all z if $\lim \dfrac{a_{n+1}}{a_n} = 0$.

Proof: By the Limit Ratio Test this if $|z|. \lim \dfrac{a_{n+1}}{a_n} = 0 < 1$, which holds for all z. ✋☺

§7.4. Binomial Coefficients

The power series that is used to define the exponential function involves factorials and the basic properties make use of the binomial theorem, so let us be reminded of these things.

If n is a natural number, ***n*!** (called **factorial *n***) is defined inductively by:

$$0! = 1;$$
$$(n + 1)! = (n + 1)\, n!$$

If $n, r \in \mathbb{Z}$ and $0 \le r \le n$ the **binomial coefficient** $\binom{n}{r}$ is defined by $\binom{\boldsymbol{n}}{\boldsymbol{r}} = \dfrac{n!}{r!\,(n-r)!}$.

Theorem 11: If $0 < r \le n$ then $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.

Proof: A very clever proof of this is to say that if we are choosing r things from $\{0, 1, \ldots, n\}$ then we either choose 0 or we don't. If we choose 0 then we have to choose the remaining $r - 1$ elements from $\{1, 2, \ldots, n\}$ and there are $\binom{n}{r-1}$ such choices. If we don't choose 0 then all r choices must be from $\{1, 2, \ldots, n\}$ and in this case there are $\binom{n}{r}$ choices. ✋☺

Unfortunately this proof won't do, since it relies on our intuition. If we are to only prove things from the axioms we can't use this interpretation of the binomial coefficient.

However a purely algebraic verification, expressing the binomial coefficients in terms of factorials, is not difficult. We leave this as an exercise.

Theorem 12 (BINOMIAL THEOREM):

If $n \in \mathbb{N}$ then $(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$.

Proof: Use induction on *n*.

The coefficient of $x^r y^{n+1-r}$ in $(x + y)(x + y)^n$ is

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r} .$$ ✋☺

§7.5. The Exponential Function

For $z \in \mathbb{C}$ define **exp(z)** $= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.$

By the Ratio Limit Test this series converges for all *z*.

The real number **e** is defined by e = exp(1) and clearly e > 1. (In fact e is approximately 2.718 but we won't assume this.)

At this stage we haven't identified exp(z) as e^z but it does seem to have the right properties.

Theorem 13: $\exp(x + y) = \exp(x).\exp(y)$

Proof: LHS = $\displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{x^s}{s!}\frac{y^{r-s}}{(r-s)!}$

$$= \sum_{r=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^s}{s!}\frac{y^t}{t!} \text{ where } t = r - s.$$ ✋☺

Corollary: $\exp(n) = e^n$ for all natural numbers n.

For the moment let's restrict our attention to exp(x) for real x. Clearly exp(0) = 1 and we've defined exp(1) = e.

Theorem 14:
(1) exp(x) > 0 for all real x;
(2) exp(x) is an increasing function of x;
Proof:
(1) For real $x > 0$ it's obvious that exp(x) > 0, but if $x < 0$ the series has alternating positive and negative terms. However by Theorem 13, exp(x).exp($-x$) = exp(0) = 1. It follows that exp(x) > 0 for negative x and hence for all real x.

(2) If $y > 0$ then $\exp(y) > 1$ and so, by Theorem 13

$$\exp(x + y) > \exp(x).$$

Hence $\exp(x)$ is an increasing function of x. ✋☺

§7.6. The Logarithm Function

We must now define **log *x*** for all positive real x. We do this in three stages:

(1) If $x > 1$, $\log x$ is defined to be $\{y \in \mathbb{Q}^+ | e^y < x\}$;

(2) If $0 < x < 1$, $\log x$ is defined to be $-\log(1/x)$;

(3) $\log 1$ is defined to be 0.

Note that $\{y \in \mathbb{Q}^+ | e^y < x\}$ is a non-empty initial segment of $\mathbb{Q}^+$ that has an upper bound and hence, by our definition, it is a positive real number.

It can then be demonstrated that $\log x$ is the inverse function to the exponential function and hence has all the usual properties. It can be represented by a power series, or at least $\log(1 + x)$ can, but this only converges for $-1 < x \leq 1$. We won't go into that here. At this moment all we need to use the log function for is to enable us to define x^y where x, y are positive reals.

We define $\boldsymbol{x^y} = e^{y.\log x}$. If m, n are both natural numbers $m^n = e^{n.\log m} = (e^{\log m})^n$ by Theorem 14. This is m^n by our previous definition so our new definition agrees with the more restricted one.

§7.7. The Trigonometric Functions

The usual definition of the trigonometric functions is essentially geometric. But as they rely on diagrams and geometric intuition these definitions are not suitable for a rigorous development. Instead we make use of the exponential function.

We have defined e^z for all complex numbers, z, by the power series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

In particular, if z is purely imaginary, we can write

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots \\ &= 1 + ix - \frac{x^2}{2!} + \frac{ix^3}{3!} - \ldots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right). \end{aligned}$$

We define $\cos x = \mathrm{Re}(e^{ix})$ and $\sin x = \mathrm{Im}(e^{ix})$, the real and imaginary parts of e^{ix} so that:

$$e^{ix} = \cos x + i \sin x.$$

This gives the power series expansions for both
$\sin x$ and $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \quad \text{and}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Also it's obvious from these power series that $\sin 0 = 0$ and $\cos 0 = 1$ and $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Moreover the sum and difference formulae are direct consequences of the fact that $e^{ix}.e^{iy} = e^{i(x+y)}$.

Theorem 15:

$$\sin(x + y) = \sin x.\cos y + \cos x.\sin y$$
$$\sin(x - y) = \sin x.\cos y - \cos x.\sin y$$
$$\cos(x + y) = \cos x.\cos y - \sin x.\sin y$$
$$\cos(x - y) = \cos x.\cos y + \sin x.\sin y$$

Proof: These come from the fact that:

$$\cos(x \pm y) \pm i \sin(x \pm y) = (\cos x \pm i \sin x)(\cos y \pm i \sin y)$$ ✋☺

Putting $x = \pm y$ we get:

$$\sin^2 x + \cos^2 x = 1;$$
$$\sin 2x = 2\sin x.\cos x \text{ and}$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

§7.8. Definition of π

Now we must define π without reference to circles and arc lengths. We define **π** by defining π/2 to be the smallest positive x for which $\cos x = 0$. Now it's not at all obvious that $\cos x = = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ will ever become zero. However we'll prove that it does so. Our proof requires the Intermediate Value Theorem and the fact that sin x is continuous. These can be proved by the usual methods and these proofs can be substantiated ultimately by the structure we've built upon the ZF axioms. So we won't repeat them here.

Theorem 16: cos $x = 0$ for some positive $x < 2$.
Proof: cos 0 = 1. But 1 – cos 2

$$= \left(\frac{2^2}{2!} - \frac{2^4}{4!}\right) + \left(\frac{2^6}{6!} - \frac{2^8}{8!}\right) + \ldots + \left(\frac{2^{4n+2}}{(4n+2)!} - \frac{2^{4n+4}}{(4n+4)!}\right) + \ldots$$

The n'th pair (for $n \geq 0$) is $\left(\frac{2^{4n+2}}{(4n+2)!} - \frac{2^{4n+4}}{(4n+4)!}\right)$

$$= \frac{2^{4n+2}}{(4n+4)!}\left[(4n+4)(4n+3) - 2^2\right] > 0.$$

At first sight, this merely gives cos 2 < 1, which is no surprise. But the first pair evaluates to $1\frac{5}{8}$ so

$$1 - \cos 2 > 1$$

which gives cos 2 < 0. By the Intermediate Value Theorem, and the fact that cos x is continuous , cos $x = 0$ for some $x > 0$.

We define $\pi/2$ to be the smallest positive x for which cos $x = 0$.

This defines π and also establishes that $\pi < 4$. ✋☺

Theorem 17: $\sin(\pi/2) = 1$.
Proof: From $\sin^2 x + \cos^2 x = 1$ we easily deduce that $\sin(\pi/2) = \pm 1$.
By taking terms in pairs we can write

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \ldots + \left(\frac{x^{4n+1}}{(4n+1)!} - \frac{x^{4n+3}}{(4n+3)!}\right) + \ldots$$

The n'th pair (for $n \geq 0$) is:

$$\frac{x^{4n+1}}{(4n+3)!}\left[(4n+3)(4n+2) - x^2\right].$$

If $0 < x < 2$ then $x^2 < 6$ and so each pair is positive. Since $\pi/2 < 2$, $\sin x > 0$. ✋☺

We can now go on to establish all the usual properties that in involve π, such as $\cos(\pi/2 - x) = \sin x$, $\cos \pi = -1$ and the fact that $\sin x$ and $\cos x$ are periodic functions with period 2π.

We are now able to express complex numbers in polar form. The **modulus** of $z = x + iy$ is defined to be $|\boldsymbol{z}| = \sqrt{x^2 + y^2}$. We can write $x = r\cos\theta$ and $y = r\sin\theta$ for some θ with $0 \leq \theta < 2\pi$. If $r \neq 0$ this representation is unique. Essentially this θ is defined to be the **argument** of z. And, since $\cos\theta$ and $\sin\theta$ have been defined to be the real and imaginary parts of $e^{i\theta}$ we can write $z = re^{i\theta}$, the **polar form** of z. In particular we get the famous identity $e^{i\pi} = -1$.

However it's more convenient to consider the argument of a non-zero complex number not as a real number but as a real number modulo 2π.

We set up the equivalence relation ~ on $\mathbb{R}$ by defining $x \sim y$ if x, y differ by an integer multiple of 2π. We let $\mathbb{R}_{2\pi}$ to be the set of equivalence classes and define $[x] + [y] = [x + y]$, where $[x]$ denotes the equivalence class containing x. It's easy to check that this operation is well-defined and associative. (In terms of group theory this is simply the quotient group $\mathbb{R}/\langle 2\pi \rangle$ where $\langle 2\pi \rangle$ is the cyclic subgroup generated by 2π.) The argument of the non-zero complex number $z = r^{i\theta}$ is defined to be $[\theta]$ in $\mathbb{R}_{2\pi}$. As a consequence we can write

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

without having to worry about what happens if we end up with a value larger than 2π.

And $\arg(-z) = \arg(z) + \arg(-1) = \arg(z) + \pi$.

For the derivatives and integrals of the trig functions we merely differentiate the power series term by term. This requires a little justification but the relevant proofs are no different, based on our ZF development, than they are normally.